MTH205: Review Problems / Final / Spring 15

Q1. Does the differential equation: $y' = y/x + \sqrt{x}$ possess a unique solution through the point $(0,0)$? Give reasons.

Q2. Find the critical points and phase portrait of $y' = y^2 - y^3$ and classify each point.

Q3. Classify each of the following differential equations as, separable, exact, linear, homogeneous, or Bernoulli. Then find the general solution of each of them.

a. $2xyy' + y^2 = 2x^2$, **b.** $y' = xy + \sqrt{y}$, **c.** $y' = \dfrac{x - 2xy}{3y^2 + x^2}$, **d.** $y' = e^{3x-2y}$

Q4. Suppose that a large mixing tank initially holds 500 gallons of water in which 50 pounds of salt have been dissolved. Water is pumped into the tank at a rate of r_1 gal/min, and then when the solution is well stirred it is pumped out at a rate of r_2 gal/min. Determine a differential equation then solve it, for the amount $x(t)$ of the salt in the tank at any time t for each of the following cases:

a. $r_1 = r_2 = 4$, and the entering water is pure.

b. $r_1 = 3$, $r_2 = 2$, and the entering water contains salt with concentration 2 lb/gal.

Q5. Initially 100 milligrams of radioactive substance were present. After 6 hours the mass had decreased by 3%. If the rate of decay is proportional to the amount of the substance present at any time, find the amount remaining after 24 hours.

Q6 Find an interval around $x = 0$ for which the initial value problem

$$\sqrt{x+1}\ y'' + \frac{1}{4-x^2} y' + y = \sin(x) \ , \quad y(0) = 1 \ , \quad y'(0) = 0$$

has a unique solution.

Q7. Find the general solution.

a. $x^3 y''' + 6x^2 y'' + 4xy' - 4y = 0$, **b.** $y^{(4)} + y'' = 0$,

c. $x^2 y'' - 4xy' + 6y = x^3$

Q8. Given that $y_1 = e^x$ is a solution of the differential equation.

$$(x+1)y'' - (x+2)y' + y = 0$$

Find the general solution.

Q9. Set up the appropriate form of the particular solution y_p , but do not determine the values of the coefficients.

$$y^{(4)} + 9y'' = 2x + (x^2 + 1)\sin(3x)$$

Q10. Use the method of undetermined coefficients to find the solution of the I.V.P.

$$y'' + 4y = 2\sin(2t) \quad , \quad y(0) = 1, y'(0) = 0$$

Q11. Use the variation of parameters method to find the general solution of the differential equation:

$$x^2 y'' - 3xy' + 4y = x^2 \ln x$$

Q12. A 32 – pound weight stretches a spring 32/5 feet. Initially the weight is released 1 foot below the equilibrium position with a downward velocity of 5 ft/sec. Determine the equation of the motion, if the surrounding medium offer a damping force numerically equal to 2 times the instantaneous velocity and the weight is driven by an external force equal to $f(t) = 12\cos 2t + 3\sin 2t$. Graph the transient and the steady-state solutions on the same coordinate axes.

Q13. A 64 – pound weight stretches a spring 0.32 foot. Initially the weight is released 2/3 foot above the equilibrium position with a downward velocity of 5 ft/sec.

- **a.** Determine the equation of the motion.
- **b.** What are the amplitude and the period of the motion?
- **c.** At what time does the mass pass through the equilibrium position for the second time?
- **d.** At what time does the mass attain its extreme displacement for the second time?

Q16. Use translation and other theorems to find:

a) $L\ \{e^{at}\sin t\ \}$ b) $L^{-1}\left\{\dfrac{s}{(s+1)^3}\right\}$ c) $L\ \{g(t)\},\ g(t) = \begin{cases} 0\ , & 0 \le t < 1 \\ t^2\ , & t \ge 1 \end{cases}$

d) $L^{-1}\ \{\dfrac{e^{-2s}}{s^3}\}$ e) $L\{t^3\cos 5t\}$

Q17. Use the Laplace transform to solve the following initial value problems

a. $y'' + y = f(t),\quad y(0) = 0,\quad y'(0) = 1$, where $f(t) = \begin{cases} 0\ , & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0\ , & t \ge 2\pi \end{cases}$

b. $y'' - 4y = e^t \quad , \quad y(0) = 1, y'(0) = 0$

c. $y'' + y = \delta(t - \pi),\ y(0) = 1, y'(0) = 0$

Q18. Use the Convolution Theorem to

a. evaluate $L\{\ t^2 * te^t\ \}$ and $L^{-1}\left\{\dfrac{1}{s^3(s-1)}\right\}$

b. solve $y' = 1 - \sin t - \int_0^t y(\tau)\ d\tau, \quad y(0) = 0$

Q19. Solve $y'' + 2y + 10y = f(t)$, $y(0) = 0, \quad y'(0) = 0,$ where f is periodic with period $T = 2\pi$ and $f(t) = \begin{cases} 1 & , 0 \le t < \pi \\ -1 & , \quad \pi \le t < 2\pi \end{cases}$

Q20. Use Laplace Transform to solve the system

$$x' = 2y + e^t, \quad x(0) = 1$$
$$y' = 8x - t, \quad y(0) = 1$$

Answers to some final review problems

Q2. critical points $y = 0, y = 1$. $y = 1$ attractor (stable). $y = 0$ semi-stable.

Q3. (a) $y' = \frac{2x^2-y^2}{2xy}$ (Homogeneous). If use substitution $u = \frac{y}{x}$, end up with $\ln|2-3u^2| = -3\ln|x| + C$. Can suppose $x > 0$. Get $y^2 = \frac{1}{3}\left(2x^2 + \frac{C}{x}\right)$.
Note this DE can also be written $y' + \frac{1}{2x}y = xy^{-1}$ (Bernoulli). Try getting same solution using Bernoulli substitution $u = y^2$. End up with linear equation $u' + \frac{1}{x}u = 2x$.

(b) $y' - xy = y^{1/2}$. Bernoulli. Set $u = y^{1/2}$. End up with $2u' - xu = 1$. Linear, $\mu(x) = e^{-x^2/4}$.

(c) $y' = f(x,y)$ with $f(tx,ty) \neq f(x,y)$ (not homogeneous). Rewrite $(2xy-x)dx+(3y^2+x^2)dy = 0$. This is exact. Potential $f(x,y) = x^2y - \frac{x^2}{2} + y^3$. Solution $f(x,y) = c$.

(d) Separable. To solve can use either the linear substitution $u = 3x - 2y$, or use separability. Easier to use separability. Answer: $\frac{1}{2}e^{2y} = \frac{1}{3}e^{3x} + c$ in implicit form.
If you use substitution $u = 3x - 2y$, you get to $\frac{u'}{3-2e^u} = 1$ which can be solved by writing this equation $\dfrac{e^{-u}}{3e^{-u}-2}du = dx$ so that $\ln|3e^{-u} - 2| = -3x + C$.
Integrating and simplifying get the same implicit equation $\frac{1}{2}e^{2y} = \frac{1}{3}e^{3x} + c$.

Q4. $A(t)$ amount of salt at time t in tank. Solve IVP $\dfrac{dA}{dt} = -\dfrac{A}{500 + (r_1 - r_2)t}$, $A(0) = 50$.

Q5. $A'(t) = -kA(t)$ with $k > 0$ (proportionality). $A(0) = 100$. $A(6) = 0.97A(0)$. Need $A(24)$.
Answer: Can solve for $A(t) = 100e^{-0.005t}$, so that $A(24) = 885$.
Can also argue (think about it) that $A(24) = (0.97)^4 A(0)$ (without any calculation).

Q6. $]-1, 2[$ (0 must be in interval). See theorem 4.1.1.

Q7. (a) Cauchy-Euler. Auxiliary polynomial $m^3 + 3m^2 - 4 = (m-1)(m^2 + 4m + 4)$. General solution $y = c_1x + c_2x^{-2} + c_3x^{-2}\ln x$.

Q8. Reduction of order: set $y = ue^x$. Get $(x+1)u'' + xu' = 0$. Solve by setting $w = u'$.
Answer: $w = C(x+2)e^{-x}$ so that $u = C_1(x+2)e^{-1} + C_2$ and $y = ux = C_1(x+2) + C_2e^x$.

Q9: $y_p = x^2(ax+b) + x\left[(c_1x^2 + c_2x + c_3)\cos 3x + (d_1x^2 + d_2x + d_3)\sin(3x)\right]$.

Q10: particular solution $y_p(t) = -\frac{t}{2}\cos(2t)$ and general solution
$y = c_1\cos 2t + c_2\sin 2t - \frac{1}{2}t\cos(2t)$. Find c_1, c_2 from initial conditions.

Q14: 0 is an ordinary point. Singular points are the roots of $(x+1)(x^2+2)$ that is $-1, \pm i\sqrt{2}$. The least distance from 0 to any of the singular points is 1. So R is at least 1.

Q16. (c) $\mathcal{L}\{g(t)\} = \mathcal{L}(t^2u(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2)\}$. Write $(t+1)^2 = t^2 + 2t + 1$ and compute.
Many of the Laplace questions were solved in class ....